A GENERAL KERNEL-BASED REGULARIZATION METHOD FOR COUPLED ELLIPTIC SINE-GORDON EQUATIONS WITH GEVREY REGULARITY

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ABSTRACT. Developments in numerical methods for problems governed by nonlinear partial differential equations underpin simulations in diverse areas. This paper is devoted to the regularization method for the coupled elliptic sine-Gordon equations. The system of equations originates from the static case of the coupled hyperbolic sine-Gordon equations modeling the coupled Josephson junctions in superconductivity, and so far it addresses the Josephson π-junctions. In general, it is ill-posed due to instability of solution. Using the modified method, we obtain stability and convergence results. The results are observed as the generalization of many previous works.

1. INTRODUCTION

The Josephson junction is a quantum mechanical device, which is formed by two superconducting electrodes separated by a very thin insulating barrier. The Josephson π-junction is a specific example of that when no external current or magnetic field is applied. It is the corner junction made of yttrium barium copper oxide (d-wave high-temperature superconductivity). The fundamental equations modeling such a junction (see Chen et al. [5]) read

$$\frac{\partial u}{\partial t} = \frac{2EV}{P}, \quad \frac{\partial u}{\partial x} = \left(\frac{2Ed}{P\omega}\right) H_2, \quad \frac{\partial u}{\partial y} = \left(-\frac{2Ed}{P\omega}\right) H_1,$$

$$J(x,y) = -J_0(x,y)\sin u,$$

where u is functional to describe the relative phase between the superconducting metals I and II, causing the Josephson tunneling current J per unit area which depends on the properties J_0 of the barrier. Experimentally, E corresponds to the electron charge, V is considered as a time-and-space-dependent potential difference across the barrier, P is a Planck's constant, d is a constant with respect to the London penetration depths for the metals, ω performs the speed of light, and H_j represents the x and y component of the magnetic field.

After substituting those equations into the Maxwell equation, it yields the barrier equation or the hyperbolic sine-Gordon equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - c_0\frac{\partial^2 u}{\partial t^2} = -\gamma^{-2}\sin u, \tag{1.1}$$

where c_0 depends only on ω and γ includes ω, P, E, d and J_0.

The static version of (1.1) thus reads

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\gamma^{-2}\sin u,$$

and if we proceed the change of variable $v(x,y) = \pi - u(x,y)$ under the homogeneous Neumann boundary condition, we are concerned with the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \gamma^{-2}\sin u,$$

which addresses us to the relation of the Josephson π-function. Mathematically speaking, the Neumann condition guarantees the equivalence between those two kinds of junctions, but the others, for example, the homogeneous Dirichlet boundary condition.

From that motivation, the purpose of this paper is to consider, as the dynamical model for the coupled Josephson junctions, the following system in two-dimensional:

$$\frac{\partial^2 u}{\partial x^2} + \alpha_1\frac{\partial^2 u}{\partial y^2} + \gamma_1\sin(\delta_{11}u + \delta_{12}v) + \sigma_{11}u + \sigma_{12}v = f_1, \tag{1.2}$$

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$$\frac{\partial^2 u}{\partial x^2} + \alpha_2 \frac{\partial^2 u}{\partial y^2} + \gamma_2 \sin\left(\delta_{21} u + \delta_{22} v\right) + \sigma_{21} u + \sigma_{22} v = f_2, \tag{1.3}$$

where $\alpha_i \in \mathbb{R}_+, \gamma_i, \delta_{ij}, \sigma_{ij} \in \mathbb{R}$ are physical constants and f_i are called forcing functions, $i, j \in \{1, 2\}$.

There are hundreds of studies on the hyperbolic version of (1.2)-(1.3). For instance, Levi et al. [10] considered the chaotic behaviors of numerical solutions for the coupled hyperbolic sine-Gordon equations with damped terms. The more general system can be regarded as the problem of identification the physical constants by Ha and Nakagiri in [6] and the answer for the question of necessary condition to gain optimality for those parameters can be found there. Up-to-date, the amount of papers which are related to those problems still increases without cease. We, however, stress that the results as far as we know for the elliptic sine-Gordon equations are very scarce. In particular, we only find some theoretical and numerical researches, e.g. in [4, 5, 11, 13] and references therein. Additionally, it should be mentioned that the Cauchy problem for such equations is ill-posed in principle in the sense of Hadamard where the stability of solution fails. Consequently, our analysis will underscore the so-called regularization method.

In recent years, various types of regularization methods have been of continuous interest to researchers in a wide range of disciplines. We refer the reader to many interesting results in [1, 3, 2, 7, 8, 12]. We notably remark the studies of Tuan and his co-authors on the nonlinear elliptic equations for which they are strongly related to our tackled equation (1.2)-(1.3). In fact, the authors considered in [16] the modified method to show that there exists a filtering kernel stabilizing the exponential instability when the spectral representation of solution is used. Only a short time ago, taking a more general nonlinear source term into account, they successfully obtained in [15] a new landmark by using the truncation approach. In light of the aforementioned works, we put ourselves into the study of the modified method to regularize the system (1.2)-(1.3).

Let $(0, a) \times (0, b)$ for $a, b > 0$, a couple of real unknown functions (u, v) is sought for $(x, y) \in [0, a] \times [0, b]$. The problem given by (1.2)-(1.3) naturally along with the zero Neumann conditions:

$$\frac{\partial u}{\partial y}(x, 0) = \frac{\partial u}{\partial y}(x, b) = \frac{\partial v}{\partial y}(x, 0) = \frac{\partial v}{\partial y}(x, b) = 0, \quad x \in (0, a), \tag{1.4}$$

and associated with initial conditions

$$\left(u(0, y), v(0, y)\right) = \left(u_0(y), v_0(y)\right), \quad \left(\frac{\partial u}{\partial x}(0, y), \frac{\partial v}{\partial x}(0, y)\right) = \left(u_1(y), v_1(y)\right), \quad y \in (0, b). \tag{1.5}$$

In this paper, we consider problem 1.2-1.5, and prove stability and convergence results of approximate solutions constructed by the generalization of modified method.

2. A theoretical framework of regularization methods by spectral theory

2.1. **Abstract settings.** Let $\langle \cdot, \cdot \rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a functional space. The notation $\|\cdot\|$ stands for the norm in L^2 and we denote by $\|\cdot\|_X$ the norm in the Banach space X. We call X' the dual space of X and denote by $C([0, a]; X)$ for the Banach space of real functions $u : (0, a) \to X$ measurable, such that

$$\|u\|_{C([0,a];X)} = \sup_{0 \le x \le a} \|u(x, \cdot)\|_X < \infty,$$

We define the usual inner product and norm of $L^2(0, b)$ by

$$\langle u, v \rangle = \int_0^b u(y) v(y) \, dy, \quad \|u\| = \langle u, u \rangle^{1/2}, \quad \forall u, v \in L^2(0, b).$$

Let us also define the functional space

$$\mathbb{V} = \left\{u \in H^1(0, b) : u_y(0) = u_y(b) = 0\right\},$$

the closed subspace of $H^1(0, b)$.

It is significant to remark that $\{\phi_n\}_{n \in \mathbb{N}} \in \mathbb{V} \cap C^\infty([0, b])$ is an orthonormal basis generated by the operator $-\frac{\partial^2}{\partial y^2}$ in $L^2(0, b)$ and it is associated with the eigenvalue $\{\lambda_n\}_{n \in \mathbb{N}}$ which is such that

$$0 < \lambda_1 < \lambda_2 < \ldots \lim_{n \to \infty} \lambda_n = \infty.$$

We turn to introduce the abstract Gevrey class of functions of order $s > 0$ and index $\nu > 0$, denoted by $\mathbb{G}_\nu^s$, defined by

$$\mathbb{G}_\nu^s = \left\{u \in L^2(0, b) : \sum_{n=1}^\infty \lambda_n^s e^{2\nu\lambda_p} |\langle u, \phi_n \rangle|^2 < \infty\right\},$$

equipped with the norm

$$\|u\|_{\mathbb{G}^s_\nu} = \left(\sum_{n=1}^{\infty} \lambda_n^s e^{2\nu\lambda_p} |\langle u, \phi_n \rangle|^2 \right)^{1/2} < \infty.$$

We now make the following assumptions:

(A_1) $u_0, v_0, u_1, v_1 \in L^2(0,b)$ and the functions $u_0^\epsilon, v_0^\epsilon, u_1^\epsilon, v_1^\epsilon \in L^2(0,b)$ are measurement datum with noise level $\epsilon > 0$ such that

$$\|u_0 - u_0^\epsilon\| \le \epsilon, \quad \|v_0 - v_0^\epsilon\| \le \epsilon, \quad \|u_1 - u_1^\epsilon\| \le \epsilon, \quad \|v_1 - v_1^\epsilon\| \le \epsilon.$$

(A_2) $f_1, f_2 \in C\left([0,a]; L^2(0,b)\right)$.

Our objective in this paper is to develop the computational foundations, we further assume that the system (1.2)-(1.5) has a unique solution in Ω. In the next subsection, we shall give the regularization method and show our main theoretical results.

2.2. **Regularization method, well-posedness and convergence.** Let (u,v) be a pair of solutions to the problem (1.2)-(1.5). So, the Fourier series corresponding to u and v are given by

$$u(x) = \sum_{n=1}^{\infty} \langle u(x), \phi_n \rangle \phi_n, \quad v(x) = \sum_{n=1}^{\infty} \langle v(x), \phi_n \rangle \phi_n.$$

Multiplying each equation of the problem by $\phi_n(y)$, and integrating with respect to y from 0 to b, we, after some computations, reduce the following system of Cauchy problem:

$$\frac{d^2}{dx^2} \langle u(x), \phi_n \rangle - \alpha_1 \lambda_n \langle u(x), \phi_n \rangle = \langle \mathcal{F}_1(x,u,v), \phi_n \rangle, \tag{2.1}$$

$$\langle u(0), \phi_n \rangle = \langle u_0, \phi_n \rangle, \quad \langle u_x(0), \phi_n \rangle = \langle u_1, \phi_n \rangle, \tag{2.2}$$

$$\frac{d^2}{dx^2} \langle v(x), \phi_n \rangle - \alpha_2 \lambda_n \langle v(x), \phi_n \rangle = \langle \mathcal{F}_2(x,u,v), \phi_n \rangle, \tag{2.3}$$

$$\langle v(0), \phi_n \rangle = \langle v_0, \phi_n \rangle, \quad \langle v_x(0), \phi_n \rangle = \langle v_1, \phi_n \rangle, \tag{2.4}$$

where the functions $\mathcal{F}_1$ and $\mathcal{F}_2$ are given by

$$\mathcal{F}_1(u,v) = f_1 - \gamma_1 \sin(\delta_{11} u + \delta_{12} v) - \sigma_{11} u - \sigma_{12} v, \tag{2.5}$$

$$\mathcal{F}_2(u,v) = f_2 - \gamma_2 \sin(\delta_{21} u + \delta_{22} v) - \sigma_{21} u - \sigma_{22} v. \tag{2.6}$$

By solving the problems (2.1)-(2.2) and (2.3)-(2.4), respectively, we say, under the nonlinear spectral theory, that u and v belonging to $C\left([0,a]; L^2(0,b)\right)$ are mild solutions to the nonlinear problem (1.2)-(1.5) if they satisfy the integral equations

$$u(x) = \sum_{n=1}^{\infty} \left[\cosh\left(\sqrt{\alpha_1 \lambda_n} x\right) \langle u_0, \phi_n \rangle + \frac{\sinh\left(\sqrt{\alpha_1 \lambda_n} x\right)}{\sqrt{\alpha_1 \lambda_n}} \langle u_1, \phi_n \rangle + \int_0^x \frac{\sinh\left(\sqrt{\alpha_1 \lambda_n}(x-\xi)\right)}{\sqrt{\alpha_1 \lambda_n}} \langle \mathcal{F}_1(\xi,u,v), \phi_n \rangle d\xi \right] \phi_n, \tag{2.7}$$

$$v(x) = \sum_{n=1}^{\infty} \left[\cosh\left(\sqrt{\alpha_2 \lambda_n} x\right) \langle v_0, \phi_n \rangle + \frac{\sinh\left(\sqrt{\alpha_2 \lambda_n} x\right)}{\sqrt{\alpha_2 \lambda_n}} \langle v_1, \phi_n \rangle + \int_0^x \frac{\sinh\left(\sqrt{\alpha_2 \lambda_n}(x-\xi)\right)}{\sqrt{\alpha_2 \lambda_n}} \langle \mathcal{F}_2(\xi,u,v), \phi_n \rangle d\xi \right] \phi_n. \tag{2.8}$$

Therefore, let us observe (2.7) and (2.8) that when $n \to \infty$, the rapid escalation of $\cosh(x)$ and $\frac{\sinh(x)}{x}$ tells us the ill-posedness of the considered problem in L^2. A small perturbation in the data, in general, described by the assumption (A_1) can arbitrarily show a huge error in the solution. Thus, performing any computation is impossible in this case and then a regularization method is required.

Now given $\beta = \beta(\epsilon)$, let us define the filtering function $\Psi_{n,k,i}(\beta, x) : (0,1) \times [0,a] \to \mathbb{R}$ for $n \in \mathbb{N}, k \ge 1, i \in \{1,2\}$ which is such that

$$\Psi_{n,k,i}(\beta, x) = \frac{e^{-\sqrt{\alpha_i \lambda_n}(a-x)}}{2\beta\sqrt{\alpha_i^k \lambda_n^k} + 2e^{-\sqrt{\alpha_i \lambda_n} a}}. \tag{2.9}$$

Prior to investigating the use of our filtering function, we prove its boundedness by the following lemmas.

Lemma 1. *Given $\beta > 0, k \geq 1$ satisfy $a^k > k\beta$ then for $x \in [0,a]$ the following inequality holds*

$$\Psi_{n,k,i}(\beta, x) \leq \frac{1}{2}(ka)^{\frac{kx}{a}} \beta^{-\frac{x}{a}} \left(\ln \left(\frac{a^k}{k\beta} \right) \right)^{-\frac{kx}{a}}, \quad \forall n \in \mathbb{N}, i \in \{1, 2\}. \tag{2.10}$$

Proof. Let us first put $g(z) = \frac{1}{\beta z^k + e^{-Rz}}$, then taking its derivative implies

$$g'(z) = \frac{Re^{-Rz} - \beta k z^{k-1}}{(\beta z^k + e^{-Rz})^2}.$$

We realize that g attains its maximum at $z = z_0$ where z_0 solves the equation $e^{-Rz_0} = \frac{k\beta}{R} z_0^{k-1}$. It follows that

$$g(z) \leq \frac{1}{\beta z_0^k + \frac{k\beta}{R} z_0^{k-1}}. \tag{2.11}$$

Moreover, by using the fact that $e^{Rz_0} \geq Rz_0$ we have

$$\frac{R}{k\beta} = z_0^{k-1} e^{Rz_0} \leq \frac{1}{R^{k-1}} e^{kRz_0},$$

which leads to

$$z_0 \geq \frac{1}{kR} \ln \left(\frac{R}{k\beta}^k \right).$$

Combining this with (2.11), we conclude that

$$g(z) \leq \frac{1}{\beta z_0^k} \leq \frac{1}{\beta} \left(\frac{kR}{\ln \left(\frac{R^k}{k\beta} \right)} \right)^k.$$

As a result, we go further to obtain the fact that for $0 \leq r \leq R$

$$e^{-rz} g(z) = \frac{e^{-rz}}{(\beta z^k + e^{-Rz})^{\frac{r}{R}} (\beta z^k + e^{-Rz})^{1-\frac{r}{R}}} \leq (g(x))^{1-\frac{r}{R}} \leq (kR)^{k\left(1-\frac{r}{R}\right)} \beta^{\frac{r}{R}-1} \left(\ln \left(\frac{R^k}{k\beta} \right) \right)^{k\left(\frac{r}{R}-1\right)}. \tag{2.12}$$

Therefore, the proof of the lemma is now straightforward due to replacing $z = \sqrt{\alpha_i \lambda_n}, r = a - x$ and $R = a$ in (2.12). $\square$

Remark 2. In lieu of considering $r = a - x$ in the last step of the proof above, we further take $r = a - x + \xi$ for $x \geq \xi$. Thus, we obtain

$$\Psi_{n,k,i}(\beta, x - \xi) = \frac{e^{-\sqrt{\alpha_i \lambda_n}(a-x+\xi)}}{2\beta \sqrt{\alpha_i^k \lambda_n^k} + 2e^{-\sqrt{\alpha_i \lambda_n} a}} \leq \frac{1}{2}(ka)^{\frac{k(x-\xi)}{a}} \beta^{\frac{\xi - x}{a}} \left(\ln \left(\frac{a^k}{k\beta} \right) \right)^{\frac{k(\xi - x)}{a}}, \quad i \in \{1, 2\}. \tag{2.13}$$

So now, we define a pair of regularized solutions under construction of the filtering function in such a way that the solutions do uniquely exist and obtain themselves the stability and convergence results. Here they are given by

$$\begin{aligned} u^\epsilon(x) &= \sum_{n=1}^\infty \left[\left(\Psi_{n,k,1}(\beta, x) + \frac{e^{-\sqrt{\alpha_1 \lambda_n} x}}{2} \right) \langle u_0^\epsilon, \phi_n \rangle + \frac{1}{\sqrt{\alpha_1 \lambda_n}} \left(\Psi_{n,k,1}(\beta, x) - \frac{e^{-\sqrt{\alpha_1 \lambda_n} x}}{2} \right) \langle u_1^\epsilon, \phi_n \rangle \right] \phi_n \\ &\quad + \sum_{n=1}^\infty \int_0^x \frac{1}{\sqrt{\alpha_1 \lambda_n}} \left(\Psi_{n,k,1}(\beta, x - \xi) - \frac{e^{-\sqrt{\alpha_1 \lambda_n}(x-\xi)}}{2} \right) \langle \mathcal{F}_1(\xi, u^\epsilon, v^\epsilon), \phi_n \rangle d\xi \phi_n, \end{aligned} \tag{2.14}$$

$$\begin{aligned} v^\epsilon(x) &= \sum_{n=1}^\infty \left[\left(\Psi_{n,k,2}(\beta, x) + \frac{e^{-\sqrt{\alpha_2 \lambda_n} x}}{2} \right) \langle v_0^\epsilon, \phi_n \rangle + \frac{1}{\sqrt{\alpha_2 \lambda_n}} \left(\Psi_{n,k,2}(\beta, x) - \frac{e^{-\sqrt{\alpha_2 \lambda_n} x}}{2} \right) \langle v_1^\epsilon, \phi_n \rangle \right] \phi_n \\ &\quad + \sum_{n=1}^\infty \int_0^x \frac{1}{\sqrt{\alpha_2 \lambda_n}} \left(\Psi_{n,k,2}(\beta, x - \xi) - \frac{e^{-\sqrt{\alpha_2 \lambda_n}(x-\xi)}}{2} \right) \langle \mathcal{F}_2(\xi, u^\epsilon, v^\epsilon), \phi_n \rangle d\xi \phi_n. \end{aligned} \tag{2.15}$$

We notably mention that if we fix $v^\epsilon \in C\left([0,a]; L^2(0,b)\right)$ in (2.14), then we obtain the integral equation $u^\epsilon(x) = \mathcal{G}(u^\epsilon)(x)$ where $\mathcal{G}$ defined by the right-hand side of (2.14) maps from $C\left([0,a]; L^2(0,b)\right)$ into itself. The boundedness of the filtering function by (2.10) and (2.13) in combination with the global Lipschitz of the functional $\mathcal{F}_1$ yields the existence and uniqueness of the solution for every $\beta(\epsilon) > 0$ by the well-known Banach fixed-point theorem. The details can be directly followed in [16]. In the same vein, we also obtain the existence and uniqueness of solution to the equation (2.15).

Let us now consider the stability of (2.14)-(2.15).

Theorem 3. *If* (u^ϵ, v^ϵ) *and* (U^ϵ, V^ϵ) *are two pairs of solutions to (2.14)-(2.15), respectively, corresponding to the initial states* $(u_0^\epsilon, u_1^\epsilon, v_0^\epsilon, v_1^\epsilon)$ *and* $(U_0^\epsilon, U_1^\epsilon, V_0^\epsilon, V_1^\epsilon)$*, then for all* $0 \le x \le a$,

$$\|u^\epsilon(x) - U^\epsilon(x)\|^2 + \|v^\epsilon(x) - V^\epsilon(x)\|^2 \le C_a (ka)^{\frac{2kx}{a}} \beta^{-\frac{2x}{a}} \left(\ln\left(\frac{a^k}{k\beta}\right)\right)^{-\frac{2kx}{a}} \sum_{i\in\{0,1\}} \left(\|u_i^\epsilon - U_i^\epsilon\|^2 + \|v_i^\epsilon - V_i^\epsilon\|^2\right),$$

where C_a *is a positive constant only depending on* $a, \alpha_1, \alpha_2, \lambda_1, \gamma_1, \gamma_2, \delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}, \sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}$.

Proof. From (2.14)-(2.15), one deduces that

$$\begin{aligned}
|\langle u^\epsilon(x) - U^\epsilon(x), \phi_n\rangle| \le & \left(\Psi_{n,k,1}(\beta, x) + \frac{e^{-\sqrt{\alpha_1\lambda_n}x}}{2}\right) |\langle u_0^\epsilon - U_0^\epsilon, \phi_n\rangle| \\
& + \frac{1}{\sqrt{\alpha_1\lambda_n}} \left|\Psi_{n,k,1}(\beta, x) - \frac{e^{-\sqrt{\alpha_1\lambda_n}x}}{2}\right| |\langle u_1^\epsilon - U_1^\epsilon, \phi_n\rangle| \\
& + \int_0^x \frac{1}{\sqrt{\alpha_1\lambda_n}} \left|\Psi_{n,k,1}(\beta, x-\xi) - \frac{e^{-\sqrt{\alpha_1\lambda_n}(x-\xi)}}{2}\right| |\langle \mathcal{F}_1(\xi, u^\epsilon, v^\epsilon) - \mathcal{F}_1(\xi, U^\epsilon, V^\epsilon), \phi_n\rangle| \, d\xi,
\end{aligned}$$

$$\begin{aligned}
|\langle v^\epsilon(x) - V^\epsilon(x), \phi_n\rangle| \le & \left(\Psi_{n,k,2}(\beta, x) + \frac{e^{-\sqrt{\alpha_2\lambda_n}x}}{2}\right) |\langle v_0^\epsilon - V_0^\epsilon, \phi_n\rangle| \\
& + \frac{1}{\sqrt{\alpha_2\lambda_n}} \left|\Psi_{n,k,2}(\beta, x) - \frac{e^{-\sqrt{\alpha_2\lambda_n}x}}{2}\right| |\langle v_1^\epsilon - V_1^\epsilon, \phi_n\rangle| \\
& + \int_0^x \frac{1}{\sqrt{\alpha_2\lambda_n}} \left|\Psi_{n,k,2}(\beta, x-\xi) - \frac{e^{-\sqrt{\alpha_2\lambda_n}(x-\xi)}}{2}\right| |\langle \mathcal{F}_2(\xi, u^\epsilon, v^\epsilon) - \mathcal{F}_2(\xi, U^\epsilon, V^\epsilon), \phi_n\rangle| \, d\xi.
\end{aligned}$$

Since we have, by (2.5)-(2.6), that

$$|\mathcal{F}_1(\xi, u^\epsilon, v^\epsilon) - \mathcal{F}_1(\xi, U^\epsilon, V^\epsilon)| \le (|\gamma_1||\delta_{11}| + |\sigma_{11}|)|u^\epsilon - U^\epsilon| + (|\gamma_1||\delta_{12}| + |\sigma_{12}|)|v^\epsilon - V^\epsilon|, \tag{2.16}$$

$$|\mathcal{F}_2(\xi, u^\epsilon, v^\epsilon) - \mathcal{F}_2(\xi, U^\epsilon, V^\epsilon)| \le (|\gamma_2||\delta_{21}| + |\sigma_{21}|)|u^\epsilon - U^\epsilon| + (|\gamma_2||\delta_{22}| + |\sigma_{22}|)|v^\epsilon - V^\epsilon|, \tag{2.17}$$

they thus give us, by Parseval's relation in accordance with the boundedness of the filtering function and the very basic inequality $(a+b)^2 \le 2(a^2+b^2)$, that

$$\begin{aligned}
d(x) \le & \frac{1}{2}\left((ka)^{\frac{2kx}{a}} \beta^{-\frac{2x}{a}} \left(\ln\left(\frac{a^k}{k\beta}\right)\right)^{-\frac{2kx}{a}} + 1\right) \left(\|u_0^\epsilon - U_0^\epsilon\|^2 + \|v_0^\epsilon - V_0^\epsilon\|^2\right) \\
& + \frac{1}{\alpha\lambda_1}\left((ka)^{\frac{2kx}{a}} \beta^{-\frac{2x}{a}} \left(\ln\left(\frac{a^k}{k\beta}\right)\right)^{-\frac{2kx}{a}} + 1\right) \left(\|u_1^\epsilon - U_1^\epsilon\|^2 + \|v_1^\epsilon - V_1^\epsilon\|^2\right) \\
& + \int_0^x \frac{C}{\alpha\lambda_1}\left((ka)^{\frac{2k(x-\xi)}{a}} \beta^{\frac{2(\xi-x)}{a}} \left(\ln\left(\frac{a^k}{k\beta}\right)\right)^{\frac{2k(\xi-x)}{a}} + 1\right) d(\xi)\, d\xi,
\end{aligned} \tag{2.18}$$

where $d(x) = \|u^\epsilon(x) - U^\epsilon(x)\|^2 + \|v^\epsilon(x) - V^\epsilon(x)\|^2$, $\alpha = \min\{\alpha_1, \alpha_2\} > 0$, $C = \sum_{i,j\in\{1,2\}} (|\gamma_i||\delta_{ij}| + |\sigma_{ij}|)^2$.

Assume that $\beta \in (0,1)$ is small enough such that $(ka)^{\frac{2kx}{a}} \beta^{-\frac{2x}{a}} \left(\ln\left(\frac{a^k}{k\beta}\right)\right)^{-\frac{2kx}{a}} \ge 1$ for all $x \in [0,a]$, it then follows from (2.15) that

$$\begin{aligned}
d(x) \le & (ka)^{\frac{2kx}{a}} \beta^{-\frac{2x}{a}} \left(\ln\left(\frac{a^k}{k\beta}\right)\right)^{-\frac{2kx}{a}} \left[\|u_0^\epsilon - U_0^\epsilon\|^2 + \|v_0^\epsilon - V_0^\epsilon\|^2 + \frac{2}{\alpha\lambda_1}\left(\|u_1^\epsilon - U_1^\epsilon\|^2 + \|v_1^\epsilon - V_1^\epsilon\|^2\right)\right] \\
& + \frac{2C}{\alpha\lambda_1}\int_0^x (ka)^{\frac{2k(x-\xi)}{a}} \beta^{\frac{2(\xi-x)}{a}} \left(\ln\left(\frac{a^k}{k\beta}\right)\right)^{\frac{2k(\xi-x)}{a}} d(\xi)\, d\xi.
\end{aligned} \tag{2.19}$$

Multiplying both sides of (2.19) by $(ka)^{-\frac{2kx}{a}}\beta^{\frac{2x}{a}}\left(\ln\left(\frac{a^k}{k\beta}\right)\right)^{\frac{2kx}{a}}$ and putting $w(x)=(ka)^{-\frac{2kx}{a}}\beta^{\frac{2x}{a}}\left(\ln\left(\frac{a^k}{k\beta}\right)\right)^{\frac{2kx}{a}}d(x)$ and thanks to Gronwall's inequality, we obtain

$$w(x)\le\left[\|u_0^\epsilon-U_0^\epsilon\|^2+\|v_0^\epsilon-V_0^\epsilon\|^2+\frac{2}{\alpha\lambda_1}\left(\|u_1^\epsilon-U_1^\epsilon\|^2+\|v_1^\epsilon-V_1^\epsilon\|^2\right)\right]\exp\left(\frac{2Cx}{\alpha\lambda_1}\right),$$

or it can be written as

$$d(x)\le(ka)^{\frac{2kx}{a}}\beta^{-\frac{2x}{a}}\left(\ln\left(\frac{a^k}{k\beta}\right)\right)^{-\frac{2kx}{a}}\left[\|u_0^\epsilon-U_0^\epsilon\|^2+\|v_0^\epsilon-V_0^\epsilon\|^2+\frac{2}{\alpha\lambda_1}\left(\|u_1^\epsilon-U_1^\epsilon\|^2+\|v_1^\epsilon-V_1^\epsilon\|^2\right)\right]\exp\left(\frac{2Cx}{\alpha\lambda_1}\right). \tag{2.20}$$

This completes the proof of the theorem. □

In order to prove the convergence, we follow the strategy that proves the convergence between the exact solutions u given by (2.7)-(2.8) and the regularized solutions corresponding the exact datum. Let us consider the following theorem.

Theorem 4. *Let us define* $(\mathcal{U}^\epsilon,\mathcal{V}^\epsilon)$ *by*

$$\begin{aligned}\mathcal{U}^\epsilon(x) &= \sum_{n=1}^\infty\left[\left(\Psi_{n,k,1}(\beta,x)+\frac{e^{-\sqrt{\alpha_1\lambda_n}x}}{2}\right)\langle u_0,\phi_n\rangle+\frac{1}{\sqrt{\alpha_1\lambda_n}}\left(\Psi_{n,k,1}(\beta,x)-\frac{e^{-\sqrt{\alpha_1\lambda_n}x}}{2}\right)\langle u_1,\phi_n\rangle\right]\phi_n\\ &\quad+\sum_{n=1}^\infty\int_0^x\frac{1}{\sqrt{\alpha_1\lambda_n}}\left(\Psi_{n,k,1}(\beta,x-\xi)-\frac{e^{-\sqrt{\alpha_1\lambda_n}(x-\xi)}}{2}\right)\langle\mathcal{F}_1(\xi,\mathcal{U}^\epsilon,\mathcal{V}^\epsilon),\phi_n\rangle d\xi\phi_n,\end{aligned} \tag{2.21}$$

$$\begin{aligned}\mathcal{V}^\epsilon(x) &= \sum_{n=1}^\infty\left[\left(\Psi_{n,k,2}(\beta,x)+\frac{e^{-\sqrt{\alpha_2\lambda_n}x}}{2}\right)\langle v_0,\phi_n\rangle+\frac{1}{\sqrt{\alpha_2\lambda_n}}\left(\Psi_{n,k,2}(\beta,x)-\frac{e^{-\sqrt{\alpha_2\lambda_n}x}}{2}\right)\langle v_1,\phi_n\rangle\right]\phi_n\\ &\quad+\sum_{n=1}^\infty\int_0^x\frac{1}{\sqrt{\alpha_2\lambda_n}}\left(\Psi_{n,k,2}(\beta,x-\xi)-\frac{e^{-\sqrt{\alpha_2\lambda_n}(x-\xi)}}{2}\right)\langle\mathcal{F}_2(\xi,\mathcal{U}^\epsilon,\mathcal{V}^\epsilon),\phi_n\rangle d\xi\phi_n.\end{aligned} \tag{2.22}$$

Suppose that the solutions given by (2.7)-(2.8) are such that $u\in C\left([0,a];\mathbb{G}_{\nu_1}^{s_1}\right),u_x\in C\left([0,a];\mathbb{G}_{\nu_1}^{s_2}\right)$ *and* $v\in C\left([0,a];\mathbb{G}_{\nu_2}^{s_1}\right),v_x\in C\left([0,a];\mathbb{G}_{\nu_2}^{s_2}\right)$ *for* $\nu_1\ge\frac{\alpha_1 a}{2},\nu_2\ge\frac{\alpha_2 a}{2}$ *and* $s_1=k,s_2=k-1$. *Then for* $\beta\in(0,1)$ *the error estimate over* $L^2(0,b)$ *between* (u,v) *and* $(\mathcal{U}^\epsilon,\mathcal{V}^\epsilon)$ *is uniformly given by*

$$\|u(x)-\mathcal{U}^\epsilon(x)\|^2+\|v(x)-\mathcal{V}^\epsilon(x)\|^2\le C_a(ka)^{\frac{2kx}{a}}\beta^{2\left(1-\frac{x}{a}\right)}\left(\ln\left(\frac{a^k}{k\beta}\right)\right)^{-\frac{2kx}{a}},$$

where C_a *is a positive constant only depending on* $a,k,\alpha_1,\alpha_2,\lambda_1,\gamma_1,\gamma_2,\delta_{11},\delta_{12},\delta_{21},\delta_{22},\sigma_{11},\sigma_{12},\sigma_{21},\sigma_{22}$.

Proof. Observe (2.7)-(2.8), it is easy to verify that

$$\langle u(x),\phi_n\rangle+\frac{\langle u_x(x),\phi_n\rangle}{\sqrt{\alpha_1\lambda_n}}=e^{\sqrt{\alpha_1\lambda_n}x}\left(\langle u_0,\phi_n\rangle+\frac{\langle u_1,\phi_n\rangle}{\sqrt{\alpha_1\lambda_n}}+\int_0^x\frac{e^{-\sqrt{\alpha_1\lambda_n}\xi}}{\sqrt{\alpha_1\lambda_n}}\langle\mathcal{F}_1(\xi,u,v),\phi_n\rangle d\xi\right),$$

$$\langle v(x),\phi_n\rangle+\frac{\langle v_x(x),\phi_n\rangle}{\sqrt{\alpha_2\lambda_n}}=e^{\sqrt{\alpha_2\lambda_n}x}\left(\langle v_0,\phi_n\rangle+\frac{\langle v_1,\phi_n\rangle}{\sqrt{\alpha_2\lambda_n}}+\int_0^x\frac{e^{-\sqrt{\alpha_2\lambda_n}\xi}}{\sqrt{\alpha_2\lambda_n}}\langle\mathcal{F}_2(\xi,u,v),\phi_n\rangle d\xi\right).$$

We put $d_1(x)=u(x)-\mathcal{U}^\epsilon(x)$ and $d_2(x)=v(x)-\mathcal{V}^\epsilon(x)$. First, we have the following

$$\begin{aligned}d_1(x) &= \sum_{n=1}^\infty\frac{\beta\sqrt{\alpha_1^k\lambda_n^k}}{2\beta\sqrt{\alpha_1^k\lambda_n^k}+2e^{-\sqrt{\alpha_1\lambda_n}x}}\left(\langle u(x),\phi_n\rangle+\frac{\langle u_x(x),\phi_n\rangle}{\sqrt{\alpha_1\lambda_n}}\right)\phi_n\\ &\quad+\sum_{n=1}^\infty\int_0^x\frac{1}{\sqrt{\alpha_1\lambda_n}}\left(\Psi_{n,k,1}(\beta,x-\xi)-\frac{e^{-\sqrt{\alpha_1\lambda_n}(x-\xi)}}{2}\right)\langle\mathcal{F}_1(\xi,u,v)-\mathcal{F}_1(\xi,\mathcal{U}^\epsilon,\mathcal{V}^\epsilon),\phi_n\rangle d\xi\phi_n\end{aligned}$$

Therefore, by Parseval's relation and (2.9) together with (2.10), (2.13) and (2.16), we get

$$\begin{aligned}
\left\| d_1(x) \right\|^2 &\le 2\beta^2 \sum_{n=1}^{\infty} \left| \Psi_{n,k,1}(\beta,x) \right|^2 e^{2\sqrt{\alpha_1\lambda_n}(a-x)} \left(\sqrt{\alpha_1^k \lambda_n^k} \left\langle u(x), \phi_n \right\rangle + \sqrt{\alpha_1^{k-1}\lambda_n^{k-1}} \left\langle u_x(x), \phi_n \right\rangle \right)^2 \\
&\quad + 2a^2 \sum_{n=1}^{\infty} \int_0^x \frac{1}{\alpha_1 \lambda_n} \left(\Psi_{n,k,1}(\beta, x-\xi) - \frac{e^{-\sqrt{\alpha_1\lambda_n}(x-\xi)}}{2} \right)^2 \left| \left\langle \mathcal{F}_1(\xi,u,v) - \mathcal{F}_1(\xi, \mathcal{U}^\epsilon, \mathcal{V}^\epsilon), \phi_n \right\rangle \right|^2 d\xi \\
&\le \beta^2 (ka)^{\frac{2kx}{a}} \beta^{-\frac{2x}{a}} \left(\ln\left(\frac{a^k}{k\beta}\right)\right)^{-\frac{2kx}{a}} \alpha_1^{k-1} \sum_{n=1}^{\infty} e^{2\sqrt{\alpha_1\lambda_n}(a-x)} \left(\alpha_1 \lambda_n^k \left| \left\langle u(x), \phi_n\right\rangle \right|^2 + \lambda_n^{k-1} \left| \left\langle u_x(x), \phi_n \right\rangle \right|^2 \right) \\
&\quad + \frac{a^2}{\alpha_1\lambda_1} \int_0^x (ka)^{\frac{k(x-\xi)}{a}} \beta^{\frac{\xi-x}{a}} \left(\ln\left(\frac{a^k}{k\beta}\right)\right)^{\frac{k(\xi-x)}{a}} \left\| \mathcal{F}_1(\xi,u,v) - \mathcal{F}_1(\xi,\mathcal{U}^\epsilon,\mathcal{V}^\epsilon) \right\|^2 d\xi \\
&\le \beta^2 (ka)^{\frac{2kx}{a}} \beta^{-\frac{2x}{a}} \left(\ln\left(\frac{a^k}{k\beta}\right)\right)^{-\frac{2kx}{a}} \alpha_1^{k-1} e^{a-x} \left(\alpha_1 \left\| u \right\|^2_{C\left([0,a];\mathbb{G}^{s_1}_{\nu_1}\right)} + \left\| u_x \right\|^2_{C\left([0,a];\mathbb{G}^{s_2}_{\nu_1}\right)} \right) \\
&\quad + \frac{2a^2 C_1}{\alpha_1\lambda_1} \int_0^x (ka)^{\frac{k(x-\xi)}{a}} \beta^{\frac{\xi-x}{a}} \left(\ln\left(\frac{a^k}{k\beta}\right)\right)^{\frac{k(\xi-x)}{a}} \left(\left\| d_1(\xi) \right\|^2 + \left\| d_2(\xi) \right\|^2 \right) d\xi, \qquad (2.23)
\end{aligned}$$

where $C_1 = \left(|\gamma_1|\,|\delta_{11}| + |\sigma_{11}|\right)^2 + \left(|\gamma_1|\,|\delta_{12}| + |\sigma_{12}|\right)^2$.

In the same technicalities, $d_2(x)$ can be estimated as follows:

$$\begin{aligned}
\left\| d_2(x) \right\|^2 &\le \beta^2 (ka)^{\frac{2kx}{a}} \beta^{-\frac{2x}{a}} \left(\ln\left(\frac{a^k}{k\beta}\right)\right)^{-\frac{2kx}{a}} \alpha_2^{k-1} e^{a-x} \left(\alpha_2 \left\| v \right\|^2_{C\left([0,a];\mathbb{G}^{s_1}_{\nu_2}\right)} + \left\| v_x \right\|^2_{C\left([0,a];\mathbb{G}^{s_2}_{\nu_2}\right)} \right) \\
&\quad + \frac{2a^2 C_2}{\alpha_2\lambda_1} \int_0^x (ka)^{\frac{k(x-\xi)}{a}} \beta^{\frac{\xi-x}{a}} \left(\ln\left(\frac{a^k}{k\beta}\right)\right)^{\frac{k(\xi-x)}{a}} \left(\left\| d_1(\xi) \right\|^2 + \left\| d_2(\xi) \right\|^2 \right) d\xi, \qquad (2.24)
\end{aligned}$$

where $C_2 = \left(|\gamma_2|\,|\delta_{21}| + |\sigma_{21}|\right)^2 + \left(|\gamma_2|\,|\delta_{22}| + |\sigma_{22}|\right)^2$.

Combining (2.23) and (2.24) gives us the fact that

$$w(x) \le \beta^2 \left(\alpha_1^{k-1} + \alpha_2^{k-1} \right) e^{a-x} C_a + \frac{2a^2}{\lambda_1} \left(\frac{C_1}{\alpha_1} + \frac{C_2}{\alpha_2} \right) \int_0^x w(\xi)\, d\xi.$$

where we have putted

$$w(x) = (ka)^{-\frac{2kx}{a}} \beta^{\frac{2x}{a}} \left(\ln\left(\frac{a^k}{k\beta}\right)\right)^{\frac{2kx}{a}} \left(\left\| d_1(x) \right\|^2 + \left\| d_2(x) \right\|^2 \right),$$

$$C_a = \alpha_1 \left\| u \right\|^2_{C\left([0,a];\mathbb{G}^{s_1}_{\nu_1}\right)} + \left\| u_x \right\|^2_{C\left([0,a];\mathbb{G}^{s_2}_{\nu_1}\right)} + \alpha_2 \left\| v \right\|^2_{C\left([0,a];\mathbb{G}^{s_1}_{\nu_2}\right)} + \left\| v_x \right\|^2_{C\left([0,a];\mathbb{G}^{s_2}_{\nu_2}\right)}.$$

By Gronwall's inequality, we thus obtain

$$\left\| d_1(x) \right\|^2 + \left\| d_2(x) \right\|^2 \le C_a \left(\alpha_1^{k-1} + \alpha_2^{k-1} \right) (ka)^{\frac{2kx}{a}} \beta^{2\left(1-\frac{x}{a}\right)} \left(\ln\left(\frac{a^k}{k\beta}\right)\right)^{-\frac{2kx}{a}} \exp\left(\frac{2a^2 x}{\lambda_1} \left(\frac{C_1}{\alpha_1} + \frac{C_2}{\alpha_2} \right) + a - x \right),$$

which leads to the proof of the theorem. □

Theorem 5. *If $\beta = \epsilon^m$ for $m \in (0,1]$, then the error estimate over $L^2(0,b)$ between the exact solutions (u,v) given by (2.7)-(2.8) and the regularized solutions (u^ϵ, v^ϵ) given by (2.14)-(2.15) is*

$$\left\| u(x) - u^\epsilon(x) \right\|^2 + \left\| v(x) - v^\epsilon(x) \right\|^2 \le C_a (ka)^{\frac{2kx}{a}} \left(\epsilon^{2m\left(1-\frac{x}{a}\right)} + \epsilon^{2\left(1-\frac{mx}{a}\right)} \right) \left(\ln\left(\frac{a^k}{k\beta}\right)\right)^{-\frac{2kx}{a}},$$

for all $0 \le x \le a$, where C_a is a positive constant only depending on $a, k, \alpha_1, \alpha_2, \lambda_1, \gamma_1, \gamma_2, \delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}, \sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}$.

Proof. The proof is straightforward by Theorem 3-4 and (A_1). In fact, choosing $\beta = \epsilon^m, m \in (0,1]$ we have

$$\begin{aligned}
\left\| u(x) - u^\epsilon(x) \right\|^2 + \left\| v(x) - v^\epsilon(x) \right\|^2 &\le 2\left(\left\| u(x) - \mathcal{U}^\epsilon(x) \right\|^2 + \left\| \mathcal{U}^\epsilon(x) - u^\epsilon(x) \right\|^2 + \left\| v(x) - \mathcal{V}^\epsilon(x) \right\|^2 + \left\| \mathcal{V}^\epsilon(x) - v^\epsilon(x) \right\|^2 \right) \\
&\le C_a (ka)^{\frac{2kx}{a}} \left(\epsilon^{2m\left(1-\frac{x}{a}\right)} + \epsilon^{2\left(1-\frac{mx}{a}\right)} \right) \left(\ln\left(\frac{a^k}{k\epsilon^m}\right)\right)^{-\frac{2kx}{a}}.
\end{aligned}$$

This completes the proof of the theorem. □

3. Concluding remarks

We have successfully extended the modified method to the system of elliptic sine-Gordon equations wherein the Gevrey regularity plays a strongly powerful position in our analysis. The method is shown to be good convergence in comparison with well-known methods, such as quasi-reversibility method and quasi-boundary value method. In fact, one of superficial advantages is described by the error estimate at $x = a$ is valid in logarithmic type, i.e.

$$\left\| u\left(a\right) - u^{\epsilon}\left(a\right)\right\|^2 + \left\| v\left(a\right) - v^{\epsilon}\left(a\right)\right\|^2 \le C_a \left(ka\right)^{2k}\left(1+\epsilon^{2(1-m)}\right)\left(\ln\left(\frac{a^k}{k\epsilon^m}\right)\right)^{-2k} \le C_a \left(\ln\left(\frac{a^k}{k\epsilon^m}\right)\right)^{-2k}, \quad m \in (0,1],$$

which cannot happen in the study of quasi-reversibility method ([9]).

Moreover, the convergence rate is quite general. In principle, it is of the order $\mathcal{O}\left(\epsilon^m\left(\ln\left(\frac{a^k}{k\epsilon^m}\right)\right)^{-\frac{kx}{a}}\right)$ and this is a generalization of many previous results, *e.g.* [16, 14].

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